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Generalised sk-Spline Interpolation on Compact Abelian Groups

J. Levesley and A. K. Kushpel

Department of Mathematics and Computer Science, University of Leicester, University Road, Leicester LE1 7RH, England

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The notion of sk-spline is generalised to arbitrary compact Abelian groups. A class of conditionally positive definite kernels on the group is identified, and a subclass corresponding to the generalised sk-spline is used for constructing interpolants, on scattered data, to continuous functions on the group. The special case of d-dimensional torus is considered and convergence rates are proved when the kernel is a product of one-dimensional kernels, and the data are gridded. © 1999 Academic Press

1. INTRODUCTION

The use of conditionally positive definite functions has recently received a lot of attention in the literature of radial basis function approximation; see, for example, Micchelli [17], Madych and Nelson [15, 16], Narcowich and Ward [23], Sun [26], Wu and Schaback [29], and Xu and Cheney [30]. In this article we apply some of the ideas in this literature to approximation on compact Abelian groups using strictly conditionally positive definite functions of order one. In Gutzmer [7], techniques for the construction of interpolants on compact groups using positive definite functions are discussed, but without analysis of errors. Here we give error estimates, for interpolation on gridded data, for functions in Sobolev classes, sets of infinitely differentiable functions, analytic functions, and entire functions.

The motivation for this lies in the desire to generalise the notion of sk-spline, used for periodic approximation, to approximation on $\mathbb{T}^d$, the d-dimensional torus. Given a 2π-periodic continuous function k and $\{x_1, x_2, ..., x_n\} \subseteq [0, 2\pi)$, an sk-spline is one of the form

$$\mathrm{sk}(x) = c_0 + \sum_{i=1}^{n} c_i k(x - x_i), \tag{1}$$

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where $\sum_{i=1}^{n} c_i = 0$. This is how interpolants are constructed using conditionally positive definite functions of order one; see [29].

The sk-spline is a recent generalisation of the more familiar polynomial splines (which are realised when k is a Bernoulli monospline of an appropriate degree) and have proved effective for computing the n-widths of certain convolution classes of functions [9, 10]. The construction in (1) has been used by many authors in the case when the kernel k possesses some sign regularity properties (like cyclic variation diminishing), as well as using Taylor's Theorem to provide some error estimates [4, 19, 20, 21, 22, 24, 27]. We emphasise that we neither impose any such sign regularity condition nor use any special properties of the polynomials in order to establish our results.

Not only are sk-splines a generalization of polynomial splines but are also a generalisation of the $\mathscr{L}$-splines of Micchelli [18]. Also, as Anselone and Laurent [2] and Atteia [3], we show that, in a more general setting, a subset of sk-spline interpolants arise as a result of minimising some semi-norm, extending the results of Holladay [8] and Ahlberg, Nielson, and Walsh [1]. In Dyn *et al.* [5] the semi-norm minimisation is exploited, as in the Euclidean case, to produce pointwise error estimates for Hermite interpolation of functions in L_2 Sobolev spaces on arbitrary manifolds. In the case of the d-dimensional torus they specialise their results to produce error estimates, for the case of interpolation, of order $n^{-(r-1)/d}$, where r is the Sobolev space smoothness. We employ a different approach and obtain error estimates for functions in a wider range of Sobolev spaces, as well as for sets of functions of infinite differentiability, measured in other norms. In the case of uniform estimates for functions in L_2 Sobolev spaces we obtain an error estimate of the form $n^{-(r-1/2)/d}$.

The main advantage in using sk-splines as opposed to polynomial splines is that if the kernel of the sk-spline is infinitely smooth then the interpolation process converges at a rate governed by the underlying smoothness of the data. A similar phenomenon is present for radial basis approximation using, for example, the multiquadric functions (see [16]). A fixed degree of polynomial spline has a maximum convergence rate regardless of the smoothness of the data.

The purpose of this paper is threefold: to prove that approximation from the sk-spline subspace of the continuous functions on a compact Abelian group is possible, to give sufficient conditions on the kernel k for existence of sk-spline interpolants, and to provide error estimates in the special case of the torus. The results of Section 2 comprise some generalisations of classical results of harmonic analysis.

The paper is organised as follows.

Section 2.1. We introduce some elementary ideas from Fourier analysis on a compact Abelian group G. We consider the density, in

$\mathbf{C}(G)$, of a particular set of functions specified by their generalised Fourier series.

Section 2.2. We give conditions for the density of a more general class of functions than sk-splines, and introduce the appropriate notion of conditional positive definiteness.

Section 2.3. sk-splines are introduced as a special case of the functions from the previous section, and the well-posedness of the interpolation problem is proved.

Section 2.4. We generalise the result of Holladay by considering certain linear operators on a particular subset of the dual of the continuous functions.

Section 3. We consider the particular example of the d-dimensional torus $\mathbb{T}^d$. We prove error estimates for interpolants, on gridded data, using a generalised sk-spline with kernel k, a product of one dimensional strictly conditionally positive definite kernels. The functions we interpolate are in the convolution class $k * U_p$, where U_p is the unit ball of L_p. In some cases the error estimates we obtain realise the n-width and are in that sense optimal.

2. COMPACT ABELIAN GROUPS

2.1. *Preliminaries*

In this section we will introduce some results from the theory of abstract harmonic analysis. For a deeper exploration of the subject see Rudin [25] or Loomis [14].

Suppose a set G has both a group structure and a topological structure, and the group multiplication with a fixed element of the group is continuous with respect to the topology. Then G is a topological group. Furthermore, if the group is compact in this topology and the multiplication is commutative then we have a compact Abelian group. There is a unique translation invariant measure on G for which the measure of G is 1. This is the normalised Haar measure μ.

A *character* of the group G is a continuous homomorphism from G onto the unit circle, and with pointwise multiplication, the set of characters is a group $\hat{G}$, which we will call the *dual group* of G. We call the identity element in this group $\hat{e}$. In what follows a crucial result is that if G is compact, $\hat{G}$ is discrete; see [25, p. 9]

To an element in $f \in L_1(G)$, the integrable functions on G, we can associate a Fourier series

$$f \sim \sum_{\chi \in \hat{G}} \alpha_\chi \chi,$$

where

$$\alpha_\chi = \int_G f(x)\, \overline{\chi(x)}\, d\mu(x).$$

To continue we require some elementary results about $\hat{G}$, which may be found in [25, pp. 7–10].

LEMMA 1.

(a) *The orthogonality relation*:

$$\int_G \chi(x)\, d\mu(x) = \begin{cases} 1, & \chi = \hat{e}, \\ 0, & \textit{otherwise}. \end{cases}$$

(b) *If e is the identity in G then $\chi(e) = 1$ for all $\chi \in \hat{G}$.*

(c) *For all $x \in G$ and $\chi \in \hat{G}$,*

$$\chi(x^{-1}) = (\chi(x))^{-1} = \chi^{-1}(x) = \overline{\chi(x)}.$$

(d) *The convolution theorem*: *Let f, $g \in L_1(G)$, with $f \sim \sum_{\chi \in \hat{G}} \alpha_\chi \chi$, and $g \in \sum_{\chi \in \hat{G}} \beta_\chi \chi$. Then $f * g \in L_1(G)$ and*

$$f * g \sim \sum_{\chi \in \hat{G}} \alpha_\chi \beta_\chi \chi.$$

2.2. G_0k-*Splines*

We begin with the definition of a more general class of functions than the sk-splines, which we will introduce in the next section as a special case.

DEFINITION 2. Let $\hat{G}_0 \subset \hat{G}$ be finite, and k be a fixed continuous function. Then a G_0k-spline is a function of the form

$$f + \sum_{i=1}^{n} c_i k(\cdot\, x_i^{-1}),$$

for some $n \in \mathbb{N}$, and $x_1, x_2, ..., x_n \in G$, where f is in the linear span of $\hat{G}_0$, and

$$\sum_{i=1}^{n} c_i \chi_0(x_i) = 0, \qquad \text{for all} \quad \chi_0 \in \hat{G}_0.$$

In this section we want to investigate the density of G_0k-splines in $\mathbf{C}(G)$. To do this we need a preliminary definition and a theorem, the proof of which can be found in [14].

DEFINITION 3. A subset A of a linear space X is fundamental if the linear span of A is dense in X.

THEOREM 4. *The set of characters is fundamental in* $\mathbf{C}(G)$.

We now state and prove the main result of this section.

THEOREM 5. *Let G be a compact, Abelian, metric group, with metric ρ, $\hat{G}_0 \subseteq \hat{G}$ be finite and $\{\chi: \bar{\chi} \in \hat{G}_0\} = \hat{G}_0$. Furthermore, let the continuous function $k = \sum_{\chi \in \hat{G} \backslash \hat{G}_0} \alpha_\chi \chi$ in $L_2(G)$, where $\alpha_\chi \neq 0$ for any $\chi \in \hat{G} \backslash \hat{G}_0$. Then, the set of G_0k-splines is fundamental in* $\mathbf{C}(G)$.

Proof. Using Theorem 4 we need only show that we can approximate any character uniformly by a function of the given form. It is clear that we can approximate any character in $\chi \in \hat{G}_0$ by setting $f = \chi$ and $c_i = 0$ for all i. So, it remains only to show that we can uniformly approximate any character $\chi \in \hat{G} \backslash \hat{G}_0$ in the required way.

Using Lemma 1 (d) we can write, for any $\chi \in \hat{G} \backslash \hat{G}_0$,

$$\begin{aligned} \chi &= \frac{1}{\alpha_\chi} k * \chi \\ &= \frac{1}{\alpha_\chi} \int_G k(\cdot\, y^{-1})\, \chi(y)\, d\mu(y). \end{aligned}$$

To prove the result we need to find an integration rule which converges for continuous functions and preserves the orthogonality of the characters. This rule will converge pointwise for each x in the last equation above, which will be uniform because G is compact. If G is a torus the characters are complex exponentials. Thus, a tensor product of rectangle rules with sufficient points will preserve the orthogonality of characters.

Let $\chi_1, \chi_2, ..., \chi_\ell$ be an enumeration of $\hat{G}_0$ and let $\chi_{\ell+1} = \chi$. By considering the homomorphism $T: G \to \mathbb{T}^{\ell+1}$ given by $T(g) = \bigoplus_{j=1}^{\ell+1} \chi_i(g)$ we see that G can be decomposed into the sum of a subgroup of $\mathbb{T}^{\ell+1}$ and $K = \ker(T)$. As we just observed, the result on the torus is trivial; let the weights and abscissae of the rule on G/K be β_i and y_i, $i \in \mathcal{I}_1$, where $\mathcal{I}_1$ is finite.

Because K is the kernel of T, it is in the kernel of each χ_i, $i = 1, 2, ..., \ell+1$. Hence, any integration rule on K which is exact for constants will preserve the orthogonality of the characters, as the characters are all constant on K. Thus, we choose a partition for $K = \bigcup_{i \in \mathcal{I}_2} \Omega_i$, where $\mathcal{I}_2$ is finite, with $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, $\operatorname{diam}(\Omega_i) \leqslant \varepsilon$, and $\mu(\Omega_i) = \beta_i(\varepsilon)$, $i \in \mathcal{I}_2$. Then, if $y_i \in \Omega_i$, $i \in \mathcal{I}_2$,

$$\left|\int_K f(y)\,d\mu(y) - \sum_{i \in \mathscr{I}} \beta_i f(y_i)\right| = \left|\sum_{i \in \mathscr{I}} \int_{\Omega_i} (f(y) - f(y_i))\,d\mu(y)\right|$$
$$\leqslant \sum_{i \in \mathscr{I}} \int_{\Omega_i} |f(y) - f(y_i)|\,d\mu(y)$$
$$\leqslant \omega(f, \varepsilon)$$

whenever the modulus of continuity $\omega(f, \varepsilon) = \sup_{\rho(x, y) \leqslant \varepsilon} |f(x) - f(y)|$ is finite. If f is continuous on G, $\omega(f, \varepsilon) \to 0$ as $\varepsilon \to 0$, so that the integration rule converges for continuous f.

If now we set $\mathscr{I} = \mathscr{I}_1 \cup \mathscr{I}_2$, and $c_i = \beta_i / \alpha_\chi \chi(y_i)$, $i \in \mathscr{I}$, then, for any $\chi_0 \in \hat{G}_0$.

$$\sum_{i \in \mathscr{I}} c_i \chi_0(y_i) = \frac{1}{\alpha_\chi} \sum_{i \in \mathscr{I}} \beta_i \chi(y_i)\,\chi_0(y_i)$$
$$= \int_G \chi\chi_0\,d\mu$$
$$= 0,$$

since $\chi_0 \neq \bar{\chi}$ due to the fact that the $\hat{G}_0$ is closed under conjugation. Also, because the integration rule converges for continuous functions,

$$\chi(x) = \frac{1}{\alpha_\chi} \int_G k(xy^{-1})\,\chi(y)\,d\mu(y)$$
$$\approx \frac{1}{\alpha_\chi} \sum_{i \in \mathscr{I}} \beta_i i \chi(y_i)\,k(xy_i^{-1})$$
$$= \sum_{i \in \mathscr{I}} c_i k(xy_i^{-1}). \quad \blacksquare$$

We now introduce the set of kernels which we will use to construct our interpolants.

Definition 6. Given a subset $\hat{G}_0 \subseteq \hat{G}$, a continuous function g is conditionally positive definite with respect to $\hat{G}_0$, which we abbreviate to $\hat{G}_0 - CPD$ if, for any $n \in \mathbb{N}$, and distinct $x_1, x_2, ..., x_n \in G$,

$$\sum_{i, j=1}^{n} c_i \bar{c}_j g(x_i x_j^{-1}) \geqslant 0,$$

for any $0 \neq c \in \mathbb{C}^n$ satisfying

$$\sum_{i=1}^{n} c_i \chi(x_i) = 0, \qquad \text{for every} \quad \chi \in \hat{G}_0.$$

If there is a strict inequality in the quadratic form above then we say that g is strictly conditionally positive definite with respect to $\hat{G}_0$, or g is $\hat{G}_0$-SCPD. If $\hat{G}_0$ is the empty set then g is (strictly) positive definite (S)PD.

EXAMPLE 7. A simple example of a positive definite function $\phi * \tilde{\phi}$, with $\phi \in L_2(G)$, where $\tilde{\phi}(x) = \bar{\phi}(x^{-1})$. For

$$\begin{aligned}
&\sum_{i,\, j=1}^{n} c_i \bar{c}_j \phi * \tilde{\phi}(x_i x_j^{-1}) \\
&\quad = \sum_{i,\, j=1}^{n} c_i \bar{c}_j \int_G \phi(x_i x_j^{-1} y^{-1})\, \tilde{\phi}(y)\, d\mu(y) \\
&\quad = \sum_{i,\, j=1}^{n} c_i \bar{c}_j \int_G \phi(x_i y^{-1})\, \tilde{\phi}(x_j y^{-1})\, d\mu(y) \\
&\quad = \int_G \left| \sum_{i=1}^{n} c_i \phi(x_i y^{-1}) \right|^2 d\mu(y) \geqslant 0,
\end{aligned}$$

where in the penultimate step above we have used the fact that the Haar measure is invariant under translation.

In the next theorem we characterise a set of $\hat{G}_0$-SCPD functions.

THEOREM 8. *Let $\hat{G}_0 \subseteq \hat{G}$. Then, if*

$$g = \sum_{\chi \in \hat{G} \backslash \hat{G}_0} \alpha_\chi \chi,$$

with $\alpha_\chi > 0$ and

$$\sum_{\chi \in \hat{G} \backslash \hat{G}_0} \alpha_\chi^{1/2} < \infty,$$

then g is $\hat{G}_0$-SCPD.

Proof. Using Lemma 1(d), we can write $g = \phi * \tilde{\phi}$, where

$$\phi = \sum_{\chi \in \hat{G} \backslash \hat{G}_0} \beta_\chi \chi,$$

setting $\beta_\chi = \alpha_\chi^{1/2}$, and then

$$\sum_{\chi \in \hat{G} \backslash \hat{G}_0} |\beta_\chi| < \infty. \tag{2}$$

Now, suppose that, given $n \in \mathbb{N}$ and $x_1, x_2, ..., x_n \in G$,

$$\sum_{i=1}^{n} c_i \chi(x_i) = 0, \qquad \text{for every} \quad \chi \in \hat{G}_0, \tag{3}$$

and

$$\begin{aligned} \sum_{i,\, j=1}^{n} c_i \bar{c}_j g(x_i x_j^{-1}) &= \int_G \left| \sum_{i=1}^{n} c_i \phi(x_i y^{-1}) \right|^2 d\mu(y) \\ &= 0. \end{aligned}$$

The computational step above follows exactly the same as in Example 7. For this to be true,

$$\sum_{i=1}^{n} c_i \phi(x_i y) = 0, \qquad \text{for all} \quad y \in G,$$

because ϕ, and hence the above sum, is continuous. Now,

$$\begin{aligned} \sum_{i=1}^{n} c_i \phi(x_i y) &= \sum_{i=1}^{n} c_i \sum_{\chi \in \hat{G} \backslash \hat{G}_0} \beta_\chi \chi(x_i y) \\ &= \sum_{\chi \in \hat{G} \backslash \hat{G}_0} \gamma_\chi \chi(y), \end{aligned}$$

with

$$\gamma_\chi = \beta_\chi \sum_{i=1}^{n} c_i \chi(x_i),$$

where we have used the multiplicative property of the characters and changed the order of summation, justified because of (2). Thus, γ_χ is zero, for each $\chi \in \hat{G}/\hat{G}_0$, which, in turn implies that

$$\sum_{i=1}^{n} c_i \chi(x_i) = 0, \qquad \text{for every} \quad \chi \in \hat{G}/\hat{G}_0,$$

as $\beta_\chi \neq 0$ for any $\chi \in \hat{G}/\hat{G}_0$. Putting this together with (3) we see that

$$\sum_{i=1}^{n} c_i \chi(x_i) = 0, \qquad \text{for every} \quad \chi \in \hat{G}.$$

Now, the characters are fundamental in $\mathbf{C}(G)$ so that the linear functional

$$\ell(f)=\sum_{i=1}^{n} c_i f(x_i), \qquad f \in \mathbf{C}(G),$$

annihilates a dense subset of $\mathbf{C}(G)$ and hence the whole of $\mathbf{C}(G)$. Thus $\ell=0$ and the result is proved. ∎

2.3. *sk-Splines*

We now-specialise to sk-splines, and prove that the interpolation problem for sk-splines is uniquely solvable if the kernel k is $\{\hat{e}\}$-SCPD.

DEFINITION 9. An sk-spline is an G_0k-spline with $G_0=\{e\}$.

Remarks 10. If we set $\hat{G}_0=\{\hat{e}\}$ in the Theorem 5 we see that sk-splines are dense in $\mathbf{C}(G)$ as long as all of the coefficients, except possibly that for $\hat{e}$, in the Fourier series for k are non-zero.

THEOREM 11. *Let k be $\{\hat{e}\}$-SCPD. Then, given $n \in \mathbb{N}$, and arbitrary data $v_1, v_2, ..., v_n \in \mathbb{C}$, there exist constants $c_0, c_1, ..., c_n$, satisfying*

$$\sum_{i=1}^{n} c_i=0,$$

such that

$$c_0+\sum_{j=1}^{n} c_j k(x_i x_j^{-1})=v_i, \qquad i=1, 2, ..., n.$$

Proof. Let $u=[1, 1, ..., 1]^T$, $c=[c_1, c_2, ..., c_n]^T$, $v=[v_1, v_2, ..., v_n]^T$, and $K_{ij}=k(x_i x_j^{-1})$, $i, j=1, 2, ..., n$. Then the proof of the theorem amounts to the proof of the nonsingularity of the matrix

$$\begin{bmatrix} u & K \\ 0 & u^T \end{bmatrix}.$$

This in turn is the same as proving that there exists no non-trivial solution to the system of equations

$$c_0 u+Kc=0, \tag{4}$$

$$u^T c=0. \tag{5}$$

If the above equations hold then, using (5), we have, premultiplying (4) by c^T,

$$c^T K c = 0,$$

and because k is $\{\hat{e}\}$-SCPD, it must be that $c = 0$. Substituting this into (4) gives $c_0 = 0$, and the result follows. ▮

Remarks 12. The above proof will be familiar to the reader conversant with the solvability of the interpolation problem with the norm or multi-quadric function; see Light and Wayne [13].

2.4. *Semi-norm Minimization*

In this section we shall generalise the result of Holladay [8] which tells us that the periodic cubic spline is the minimiser, over all periodic interpolants, of the semi-norm

$$\left(\int_0^{2\pi} \left(\frac{d^2 g}{dx^2} \right)^2 dx \right)^{1/2}.$$

To do this we need to introduce generalised functions on the group G, and we do this, in the spirit of Gorbacuk and Gorbacuk [6], via the character group.

Let $\langle \hat{G} \rangle$ denote the linear span of $\hat{G}$. A sequence $f_m \to f$ as $m \to \infty$ in $\langle \hat{G} \rangle$ if and only if $\alpha_\chi(f_m) \to \alpha_\chi(f)$ as $m \to \infty$, for all $\chi \in \hat{G}$.

DEFINITION 13. A generalised function is a continuous linear functional on $\langle \hat{G} \rangle$. We say that a sequence of generalised functions f_m converges to a generalised function f if and only if $f_m(\chi) \to f(\chi)$ as $m \to \infty$ for all $\chi \in \hat{G}$.

We can associate with any generalised function f a Fourier series

$$f \sim \sum_{\chi \in \hat{G}} \alpha_\chi \chi,$$

where $\alpha_\chi = f(\chi)$. For example, δ, the unit in the convolution ring $L_1(G)$, has Fourier series

$$\delta \sim \sum_{\chi \in \hat{G}} \chi.$$

To prove the result that follows we need a continuous δ sequence

$$\delta_m = \sum_{\chi \in \hat{G}_m} \chi,$$

where $\{\hat{G}_m\}_{m\in\mathbb{N}}$ is a sequence of finite subsets of $\hat{G}$ satisfying

$$\hat{G}_m \subseteq \hat{G}_{m+1}, \qquad m\in\mathbb{N}, \tag{6}$$

$$\bigcup_{m=1}^{\infty} \hat{G}_m = \hat{G}, \tag{7}$$

where we recall that $\hat{G}$ is countable.

THEOREM 14. *Let $(\mathbf{C}(G))'$ be the dual space of $\mathbf{C}(G)$, and B: $L_2(g)\to L_2(G)$ be a continuous linear operator from $(\mathbf{C}(G))'$ to $\mathbf{C}(G)$. Let B^* be the adjoint of B. Suppose also that $(B^*)^{-1}$ exists and has kernel $\langle \hat{G}_0\rangle$ for some $\hat{G}_0\subseteq\hat{G}$. Furthermore, suppose that, given $\{x_i\}_{1\leqslant i\leqslant n}\subseteq G$ and complex data $\{v_i\}_{1\leqslant i\leqslant n}$,*

$$V=\left\{f\in B^*(L_2(G)) : f\in \sum_{\chi\in\hat{G}} \alpha_\chi(f)\,\chi \text{ and } f(x_i)=v_i,\ 1\leqslant i\leqslant n\right\}.$$

Then, if there is an element of V of the form

$$g(x)=g_0(x)+B^*B\left(\sum_{i=1}^{n} c_i\delta(\cdot\, x_i^{-1})\right), \qquad g_0\in\langle\hat{G}_0\rangle,$$

then g minimises, over all $f\in V$, the semi-norm

$$\left(\int_G |(B^*)^{-1}\, g(x)|^2\, d\mu(x)\right)^{1/2}.$$

Proof. Let $f\in V$. It is easy to show that

$$\begin{aligned}\int_G &|(B^*)^{-1}\, f(x)|^2\, d\mu(x)\\ &=\int_G |(B^*)^{-1}\, g(x)|^2\, d\mu(x)+\int_G |(B^*)^{-1}[f-g](x)|^2\, d\mu(x)\\ &\quad+2\Re\left(\int_G (B^*)^{-1}\, g(x)\overline{(B^*)^{-1}[f-g](x)}\, d\mu(x)\right),\end{aligned}$$

and the result follows if the last term in the right hand side above is zero.

Now,

$$\int_G (B^*)^{-1}\, g(x)\overline{(B^*)^{-1}[f-g](x)}\, d\mu(x)$$

$$= \int_G \left\{ (B^*)^{-1}\left[g_0(x) + B^*B\left[\sum_{i=1}^{n} c_i \delta(xx_i^{-1}) \right]\right]\right\} \overline{(B^*)^{-1}[f-g](x)}\, d\mu(x)$$

$$= \int_G B\left[\sum_{i=1}^{n} c_i \delta(xx_i^{-1}) \right] \overline{(B^*)^{-1}[f-g](x)}\, d\mu(x)$$

as $g_0 \in \ker (B^*)^{-1}$. Let $\{\delta_m\}_{m\in\mathbb{N}}$ be a delta sequence with respect to the sequence $\{\hat{G}_m\}_{m\in\mathbb{N}}$ satisfying equations (6) and (7) above. Then, because B is continuous,

$$\int_G (B^*)^{-1}\, g(x)\overline{(B^*)^{-1}[f-g](x)}\, d\mu(x)$$

$$= \lim_{m\to\infty} \int_G B\left[\sum_{i=1}^{n} c_i \delta_m(xx_i^{-1}) \right] \overline{(B^*)^{-1}[f-g](x)}\, d\mu(x)$$

$$= \lim_{m\to\infty} \int_G \sum_{i=1}^{n} c_i \sum_{\chi\in\hat{G}_m} \chi(xx_i^{-1})(\overline{f(x)-g(x)})\, d\mu(x)$$

$$= \lim_{m\to\infty} \int_G \sum_{i=1}^{n} c_i \sum_{\chi\in\hat{G}_m} \chi(x)\,\chi(x_i^{-1})(\overline{f(x)-g(x)})\, d\mu(x)$$

$$= \lim_{m\to\infty} \int_G \sum_{i=1}^{n} c_i \sum_{\chi\in\hat{G}_m} (\alpha_\chi(\bar{f})-\alpha_\chi(\bar{g}))\,\overline{\chi(x_i)}$$

$$= \sum_{i=1}^{n} c_i(\overline{f(x_i)-g(x_i)})$$

$$= 0.$$

as f and g interpolate the same data, where, in the above, we have used the facts that $\chi(x^{-1}) = \overline{\chi(x)}$ (Lemma 1(c)) and f and g are equal to their Fourier series. ∎

EXAMPLE 15. In this example we see that the result of Holladay is a special case of the above, when G is the unit circle. Let

$$D_r(x) = \sum_{\ell=1}^{\infty} \ell^{-r} \cos(\ell x - r\pi/2),$$

be the degree r Bernoulli monospline. For $g \in L_2(G)$ let

$$Bg = D_2 * g.$$

Then B is self adjoint, and continuous from $(\mathbf{C}(G))' \to \mathbf{C}(G)$, as it is (up to a constant) the same as two integrations. Thus $(B^*)^{-1}$ is just two differentiations, and

$$V = \left\{ f \colon \frac{d^2 f}{dx^2} \in L_2(G) \right\}.$$

Now, $B^* B = D_4$, so that the interpolants we are considering are of the form $c_0 + D_4 * (\sum_{i=1}^n c_i \delta(\cdot - x_i)) = c_0 + \sum_{i=1}^n c_i D_4(x - x_i)$ with $0 \leqslant x_1 < x_2 < \cdots x_n < 2\pi$. It is easy to show that all periodic cubic splines can be written in this form. So, the above theorem says that the cubic spline interpolant to given data at $\{x_i\}_{1 \leqslant i \leqslant n}$ is the minimiser over all interpolants with square integrable second derivatives, of the semi-norm

$$\left(\int_0^{2\pi} (f^{(2)}(x))^2 \, dx \right)^{1/2}.$$

3. INTERPOLATION ON THE TORUS

Let $\mathbb{T}^d = [0, 2\pi)^d$ be the d-dimensional torus. We shall be interpolating functions in convolution classes

$$k * U_p = \{ f = k * \phi : \phi \in U_p \},$$

where

$$U_p = U_p(\mathbb{T}^d) = \{ \phi \in L_p(\mathbb{T}^d) : \|\phi\|_p \leqslant 1 \},$$

and

$$k(x) = \prod_{m=1}^{d} k_m(x_m), \qquad x = (x_1, x_2, ..., x_d),$$

is a product of one dimensional kernels. Thus, by choosing each k_m to have a certain amount of smoothness, we can approximate functions which have different amounts of smoothness in different directions.

If we define $a_{\mathbf{z}}$, $\mathbf{z} = (z_1, z_2, ..., z_d) \in \mathbb{Z}^d$, by

$$a_{\mathbf{z}} = \begin{cases} 0, & z_m = 0, \quad \text{for any} \quad 1 \leqslant m \leqslant d, \\ 2^{-d} \prod_{m=1}^{d} a_{|z_m|, m}, & \text{otherwise}, \end{cases}$$

then we can write $k(x)=\sum_{\mathbf{z}\in\mathbb{Z}^d} a_{\mathbf{z}} e^{i\mathbf{z}\mathbf{x}}$, where $\mathbf{zx}$ denotes the usual inner product of $\mathbf{z}$ and $\mathbf{x}$.

In the remainder of this paper C will denote a generic constant whose value is not necessarily the same at each occurrence.

Before we consider the construction of the sk-spline interpolant in more detail we give some examples in order to clarify the sorts of functions we are interpolating.

EXAMPLE 16. Suppose that

$$k_m(x_m)=\sum_{\ell=1}^{\infty} a_{\ell,m}\cos\left(\ell x_m-\frac{\beta_m\pi}{2}\right).$$

Then, for particular choices of the $a_{\ell,m}$ and the constant β_m we obtain the following concrete examples.

1. $a_{\ell,m}=\ell^{-r_m}$, $\beta_m=r_m\in\mathbb{N}_0$ gives us a Bernoulli monospline in each direction, and in this case $k*U_p$ is the unisotropic Sobolev class of functions whose $r=(r_1, r_2, ..., r_d)$th derivative is in U_p.

2. $a_{\ell,m}=\exp(-\alpha_m\ell^{rm})$, with $\alpha_m>0$, $0<r_m<1$, and $\beta_m=0$. The convolution class in this case contains infinitely differentiable functions.

3. $a_{\ell,m}=2(\rho_m)^{\ell}$, with $0<\rho_m<1$, and $\beta_m=0$, is (up to an added constant) the Poisson kernel and the convolution class consists of functions in which are analytic in each of the coordinate directions.

4. $a_{\ell,m}=\exp(-\alpha_m\ell^{r_m})$, with $\alpha_m>0, r_m>1$, and $\beta_m=0$, and the resulting convolution is an entire function of x_m.

In what follows we will be investigating the rate of convergence of interpolants (which we will describe below) on the grid $\Delta_{\mathbf{n}}$ with points

$$x_{\mathbf{j}}=(x_{\mathbf{j},1}, x_{\mathbf{j},2}, ..., x_{\mathbf{j},d}), \qquad \mathbf{j}\leqslant 2\mathbf{n}$$

where $x_{\mathbf{j},m}=j_m\pi/n_m$, and $\mathbf{j}$ and $\mathbf{n}$ are multi-indices.

In [9], for the case $d=1$, interpolants of the form

$$\mathrm{sk}(x)=c_0+\sum_{i=1}^{2n} c_j k(x-x_j)$$

are examined where $\sum_{i=1}^{2n} c_i=0$ and $k(x)=\sum_{\ell\geqslant 1} a_\ell\cos\ell x$, with $a_\ell> a_{\ell+1}>0$ for all $\ell\in\mathbb{N}$. These form a subset of the sk-splines. Writing $\cos\ell x=(e^{i\ell x}+e^{-i\ell x})/2$ and noting that the dual group of $\mathbb{T}$ is $\{e^{i\ell x}, \ell\in\mathbb{Z}\}$, we see by Theorem 8 that k is $\{\hat{e}\}$-SCPD, and hence, by Theorem 11, that, for each m, there is a unique sk-spline interpolant to arbitrary data on $\{x_{\mathbf{j},m}\colon 1\leqslant j_m\leqslant 2n_m\}$.

Let

$$\widetilde{\mathrm{sk}}_m(x_m) = \begin{cases} 1, & x_m = 0, \\ 0, & x_m = \dfrac{j\pi}{n_m}, \qquad j = 1, 2, ..., 2n_m - 1, \end{cases}$$

be the cardinal sk-spline associated with the mth coordinate. We construct a cardinal generalised sk-spline on the grid $\Delta_{\mathbf{n}}$ with products of such one dimensional cardinal splines:

$$\widetilde{\mathrm{sk}}(\mathbf{x}) = \prod_{m=1}^{d} \widetilde{\mathrm{sk}}_m(x_m).$$

So, using the shifted cardinal sk-spline

$$\widetilde{\mathrm{sk}}(\mathbf{x} - \mathbf{x_j}) = \prod_{m=1}^{d} \widetilde{\mathrm{sk}}_m(x_m - x_{\mathbf{j},m}),$$

where $x_{\mathbf{j},m}$ is the mth component of $\mathbf{x_j}$, we can write the generalised sk-spline interpolant to $f \in k * U_p$ in the form

$$\mathrm{sk}(f, \Delta_{\mathbf{n}})(\mathbf{x}) = \sum_{\mathbf{1} \leqslant \mathbf{j} \leqslant \mathbf{2n}} f(\mathbf{x_j}) \, \widetilde{\mathrm{sk}}(\mathbf{x} - \mathbf{x_j}),$$

where $\mathbf{1} = (1, 1, ..., 1)$.

In order to prove our main theorem, Theorem 21, we need some preliminary lemmas.

LEMMA 17. *Let* $k(x) = \sum_{\ell=1}^{\infty} a_\ell \cos(\ell x)$ *uniformly, with* $a_\ell > a_{\ell+1} > 0$ *for all* $l \in \mathbb{N}$. *Then,*

(a) *for any* $n \in \mathbb{N}$, *the cardinal sk-spline at* $l\pi/n$, $l = 0, 1, ..., 2n-1$, *exists and can be written in the form*

$$\overline{\mathrm{sk}}(x) = \frac{1}{2n} + \frac{1}{2n} \sum_{l=1}^{2n-1} \frac{\rho_\ell(x)}{\rho_\ell(0)},$$

where $\rho_\ell(x) = \Re \lambda_\ell(x)$, *and*

$$\lambda_\ell(x) = \sum_{j=1}^{2n} e^{i\pi j\ell/n} \, k\left(x - \frac{j\pi}{n}\right),$$

(b) $\rho_\ell(x) = \sum_{m=1}^{\infty} \{a_{2mn-\ell} \cos((2mn - \ell)\, x) + a_{2mn+\ell} \cos((2mn + \ell)\, x)\}$
$+ a_\ell \cos(\ell x),$

(c) $\rho_\ell(x) = \rho_{2n-\ell}(x)$, $1 \leqslant \ell \leqslant n$,

(d) *if* $\sigma_\ell(x) = \Im \lambda_\ell(x)$ *then* $\sigma_\ell(x) = -\sigma_{2n-\ell}(x)$,

(e) *for all* $\ell \in \mathbb{N}$,

$$\sum_{j=1}^{2n} e^{i\pi\ell_j/n}\, \overline{\mathrm{sk}}\left(x - \frac{j\pi}{n}\right) = \frac{\lambda_\ell(x)}{\rho_\ell(0)}.$$

Proof. For proofs of (a), (b), (c), and (d) see [9, 10]. We prove only the real part of (e) here. The imaginary part follows in the same way. Set $x_j = j\pi/n$, $j = 0, 1, ..., 2n-1$. Note that we need only prove the result for $0 \leqslant j \leqslant 2n-1$ because both the left hand side and right hand side in (e) are invariant if j is altered by a multiple of $2n$.

A simple direct calculation shows that

$$\rho_\ell(x - x_j) = \rho_\ell(x)\cos(\ell x_j) + \sigma_\ell(x)\sin(\ell x_j).$$

Then, using part (a) of this lemma we have, for $\ell \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{j=1}^{2n} \cos(\ell x_j)\, \overline{\mathrm{sk}}(x - x_j) \\
&\quad = \sum_{j=1}^{2n} \cos(\ell x_j) \left\{ \frac{1}{2n} + \frac{1}{2n} \sum_{k=1}^{2n-1} \frac{\rho_k(x - x_l)}{\rho_k(0)} \right\} \\
&\quad = \frac{1}{2n} \sum_{j=1}^{2n} \cos(\ell x_j) + \frac{1}{2n} \sum_{k=1}^{2n-1} (\rho_k(0))^{-1} \\
&\qquad \times \left\{ \rho_k(x) \sum_{j=1}^{2n} \cos(\ell x_j)\cos(k x_j) + \sigma_k(x) \sum_{j=1}^{2n} \cos(\ell x_j)\sin(k x_j) \right\}.
\end{aligned}$$

Now, using the discrete orthogonality relations

$$\sum_{j=1}^{2n} \cos(\ell x_j)\cos(k x_j) = \begin{cases} 1, & k \equiv \pm\ell \equiv n \quad \text{or} \quad 0 \bmod (2n), \\ \frac{1}{2}, & k \equiv \pm\ell \not\equiv n \quad \text{or} \quad 0 \bmod (2n), \\ 0, & \text{otherwise}, \end{cases}$$

$$\sum_{j=1}^{2n} \cos(\ell x_j)\sin(k x_j) = 0, \qquad \text{for all } k, \ell,$$

we have, in mind of (d) above, that

$$\sum_{j=1}^{2n} \cos(\ell x_j)\, \overline{\mathrm{sk}}(x - x_j) = \frac{\rho_\ell(x)}{\rho_\ell(0)}, \qquad 1 \leqslant \ell \leqslant 2n-1,$$

and the result is proved. ∎

LEMMA 18. *Let $\mathbf{z} \in \mathbb{Z}^d$. Then, for any $\mathbf{x} \in \mathbb{T}^d$,*

$$\left| e^{i\mathbf{z}\mathbf{x}} - \sum_{\mathbf{j} \leqslant 2\mathbf{n}} e^{i\mathbf{z}\mathbf{x}_\mathbf{j}} \overline{\mathrm{sk}}(\mathbf{x} - \mathbf{x}_\mathbf{j}) \right| \leqslant (2\pi)^{d-1} \sum_{m=1}^{d} |\theta^m_{n_m, z_m}(x_m)|,$$

where, for $r \in \mathbb{N}$ and $s \in \mathbb{Z}$,

$$\theta^m_{r,s}(t) = e^{ist} - \sum_{i=1}^{2r} e^{i\pi ls/r} \overline{\mathrm{sk}}_m \left(t - \frac{l\pi}{r} \right).$$

Proof. First,

$$\begin{aligned} \sum_{\mathbf{j} \leqslant 2\mathbf{n}} e^{i\mathbf{z}\mathbf{x}_\mathbf{j}} \overline{\mathrm{sk}}(\mathbf{x} - \mathbf{x}_\mathbf{j}) &= \sum_{\mathbf{j} \leqslant 2\mathbf{n}} e^{i\mathbf{z}\mathbf{x}_\mathbf{j}} \prod_{m=1}^{d} \overline{\mathrm{sk}}_m(x_m - x_{\mathbf{j},m}) \\ &= \sum_{\mathbf{j} \leqslant 2\mathbf{n}} \prod_{m=1}^{d} e^{iz_m x_{\mathbf{j},m}} \overline{\mathrm{sk}}_m(x_m - x_{\mathbf{j},m}) \qquad (8) \\ &= \prod_{m=1}^{d} \sum_{\mathbf{j} \leqslant 2\mathbf{n}} e^{iz_m x_{\mathbf{j},m}} \overline{\mathrm{sk}}_m(x_m - x_{\mathbf{j},m}). \end{aligned}$$

Now, it is easy to show by induction that

$$\prod_{m=1}^{d} y_{1,m} - \prod_{m=1}^{d} y_{2,m} = \sum_{m=1}^{d} (y_{1,m} - y_{2,m}) \prod_{r=1}^{m-1} y_{2,r} \prod_{r=m+1}^{d} y_{1,r},$$

so that, if $0 \leqslant y_{1,m}, y_{2,m} < 2\pi$, $m = 1, 2, ..., d$,

$$\left| \prod_{m=1}^{d} y_{1,m} - \prod_{m=1}^{d} y_{2,m} \right| \leqslant (2\pi)^{d-1} \sum_{m=1}^{d} |y_{1,m} - y_{2,m}|.$$

Thus, using the last equation and (8), we have,

$$\begin{aligned} &\left| e^{i\mathbf{z}\mathbf{x}} - \sum_{\mathbf{j} \leqslant 2\mathbf{n}} e^{i\mathbf{z}x_\mathbf{j}} \overline{\mathrm{sk}}(\mathbf{x} - \mathbf{x}_\mathbf{j}) \right| \\ &\quad = \left| \prod_{m=1}^{d} e^{iz_m x_m} - \prod_{m=1}^{d} \sum_{j_m \leqslant 2n_m} e^{iz_m x_{\mathbf{j},m}} \overline{\mathrm{sk}}(x_m - x_{\mathbf{j},m}) \right| \\ &\quad \leqslant (2\pi)^{d-1} \sum_{m=1}^{d} \left| e^{iz_m x_m} - \sum_{j_m \leqslant 2n_m} e^{iz_m x_{\mathbf{j},m}} \overline{\mathrm{sk}}_m(x_m - x_{\mathbf{j},m}) \right| \\ &\quad = (2\pi)^{d-1} \sum_{m=1}^{d} |\theta^m_{n_m, z_m}(x_m)|. \quad \blacksquare \end{aligned}$$

LEMMA 19. *Let $k(x)=\sum_{\ell=1}^{\infty} a_\ell \cos \ell x$, where $a_\ell > a_{\ell+1} > 0$ for all $\ell \in \mathbb{N}$. Further, suppose that for $1 \leqslant |\ell| \leqslant n-1$,*

$$\sum_{m=1}^{\infty} a_{2mn-|\ell|} < C a_{2n-|\ell|}, \tag{9}$$

for every choice of $n \in \mathbb{N}$, where C is independent of n and ℓ. Then,

$$\begin{aligned} |\theta_{n,\ell}(x)| &= \left| e^{i\ell x} - \sum_{j=1}^{2n} e^{i\pi\ell j/n} \overline{\mathrm{sk}}\left(x - \frac{j\pi}{n}\right) \right| \\ &\leqslant \begin{cases} 8C \dfrac{a_{2n-|\ell|}}{a_{|\ell|}}, & 1 \leqslant |\ell| \leqslant n-1, \\ 4, & |\ell| \geqslant n. \end{cases} \end{aligned}$$

Proof. Let

$$\mu_{n,\ell}(x) = \cos(\ell x) - \sum_{j=1}^{2n} \cos\left(\frac{j\pi}{b}\right) \overline{\mathrm{sk}}\left(x - \frac{j\pi}{n}\right),$$

be the real part of $\theta_{n,\ell}$. We prove the result for $\mu_{n,\ell}$. The proof for the imaginary part of $\theta_{n,\ell}$ follows in the same way. Using Lemma 17(b) and (e) we have

$$\begin{aligned} \mu_{n,\ell}(x) &= \cos(\ell x) - \frac{\rho_\ell(x)}{\rho_\ell(0)} \\ &= \cos(\ell x) - \frac{s(x) + a_{|\ell|}\cos(\ell x)}{s(0) + a_{|\ell|}}, \end{aligned}$$

where

$$s(x) = \sum_{m=1}^{\infty} \{a_{2mn-\ell} \cos((2mn-\ell)\, x) + a_{2mn+\ell} \cos((2mn+\ell)\, x)\}.$$

Hence

$$\begin{aligned} |\mu_{n,\ell}(x)| &= \left| \frac{s(0)\cos(\ell x) - s(x)}{s(0) + a_{|\ell|}} \right| \\ &\leqslant 2, \end{aligned}$$

for all $\ell \in \mathbb{Z}$, because $s(x) \leqslant s(0)$ for all $x \in [0, 2\pi)$, and $a_{|\ell|} > 0$. However, if $1 \leqslant |\ell| \leqslant n-1$ we can make the tighter bound,

$$\begin{aligned}|\mu_{n,\ell}(x)| &\leqslant \frac{2s(0)}{a_{|\ell|}}\\ &\leqslant 4\,\frac{\sum_{m=1}^{\infty} a_{2mn-|\ell|}}{a_{|\ell|}}\\ &\leqslant 4\,\frac{Ca_{2n-|\ell|}}{a_{|\ell|}},\end{aligned}$$

using (9), where the penultimate step follows because $a_{2mn-|\ell|} > a_{2mn+|\ell|}$. ∎

LEMMA 20. *For* $p \geqslant 1$,

$$\sum_{\mathbf{z}\in\mathbb{Z}^d} a_{\mathbf{z}}^p \left(\sum_{m=1}^{d} |\theta^m_{n_m, z_m}(x_m)|\right)^p \leqslant C \sup_{1\leqslant m\leqslant d} \sum_{\ell \geqslant n_m} a^p_{\ell, m}.$$

Proof. We proceed by induction on the dimension d. If $d=1$,

$$\begin{aligned}&\sum_{\mathbf{z}\in\mathbb{Z}} a_{\mathbf{z}}^p |\theta_{n,z_1}(x)|^p\\ &\quad= \sum_{0<|\ell|<n} a^p_{|\ell|,1} |\theta_{n,\ell}(x)|^p + \sum_{|\ell|\geqslant n} a^p_{|\ell|,\mathbf{1}} |\theta_{n,\ell}(x)|^p\\ &\quad\leqslant C\left(\sum_{0<|\ell|<n} a^p_{|\ell|,\mathbf{1}} \frac{a^p_{2n-|\ell|,\mathbf{1}}}{a^p_{|\ell|,\mathbf{1}}} + \sum_{\ell\geqslant n} a^p_{\ell,\mathbf{1}}\right)\\ &\quad\leqslant C \sum_{\ell\geqslant n} a^p_{\ell,\mathbf{1}},\end{aligned}$$

where in the last two steps above we have used Lemma 19 and the fact that $a_{k,1}$ is a decreasing positive sequence.

Now, using Minkowski's inequality, we have

$$\begin{aligned}&\sum_{\mathbf{z}\in\mathbb{Z}^d} a_{\mathbf{z}}^p \left(\sum_{m=1}^{d} |\theta_{n_m,z_m}(x_m)|\right)^p\\ &\quad\leqslant C \sum_{\ell\in\mathbb{Z}} a^p_{|\ell|,d} \sum_{\mathbf{z}\in\mathbb{Z}^{d-1}} a^p_{\mathbf{z}} \left(\sum_{m=1}^{d-1} |\theta^m_{n_m,z_m}(x_m)|\right)^p\\ &\qquad + C \sum_{\ell\in\mathbb{Z}^{d-1}} a^p_{\mathbf{z}} \sum_{\ell\in\mathbb{Z}} a^p_{|\ell|,d} |\theta^d_{n_d,\ell}(x_d)|^p\\ &\quad\leqslant C \sup_{1\leqslant m\leqslant d-1} \sum_{\ell\geqslant n_m} a^p_{\ell,m} + C \sum_{\ell\geqslant n_d} a^p_{d,\ell}\\ &\quad\leqslant C \sup_{1\leqslant m\leqslant d} \sum_{\ell\geqslant n_m} a^p_{\ell,m},\end{aligned}$$

where the penultimate step follows by the inductive hypothesis and because the sequence $\{a_{\mathbf{z}}\}_{\mathbf{z}\in\mathbb{Z}^d}$ is absolutely summable. ∎

THEOREM 21. *Let $k\in\mathbf{C}(\mathbb{T}^d)$ have the absolutely convergent Fourier series*

$$k(x)=\prod_{m=1}^{d} k_m(x_m)=\prod_{m=1}^{d}\sum_{\ell=1}^{\infty} a_{\ell,m}\cos(\ell x_m),$$

where $a_{\ell,m}>a_{\ell+1,m}>0$, for all $\ell\in\mathbb{N}$, $1\leqslant m\leqslant d$, and, for $|\ell|\leqslant n_m$,

$$\sum_{s=1}^{\infty} a_{2n_m s-\ell,m}\leqslant C a_{2n_m-\ell,m},$$

where C is independent of m,ℓ, and $\mathbf{n}$. Then, for $1\leqslant p\leqslant 2\leqslant q\leqslant\infty$, with $p^{-1}-q^{-1}\geqslant 2^{-1}$,

$$\sup_{f\in k*U_{\mathbf{p}}}\|f-\mathrm{sk}(f,\Delta_{\mathbf{n}})\|_q\leqslant\max_{1\leqslant m\leqslant d}\left(\sum_{\ell\geqslant n_m}(a_{\ell,m})^{qp(q-p)^{-1}}\right)^{p^{-1}-q^{-1}}.$$

Proof. Using the fact that $f\in k*U_p$, we can write, for some $\phi\in U_p$,

$$\begin{aligned}&|f(\mathbf{x})-\mathrm{sk}(f,\delta_{\mathbf{n}})(\mathbf{x})|\\&\quad=\left|\int_{\mathbb{T}^d}k(\mathbf{x}-\mathbf{y})\,\phi(\mathbf{y})\,d\mathbf{y}-\int_{\mathbb{T}^d}\left\{\sum_{\mathbf{j}\leqslant 2\mathbf{n}}k(\mathbf{x}_{\mathbf{j}}-\mathbf{y})\,\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x}_{\mathbf{j}})\right\}\phi(\mathbf{y})\,d\mathbf{y}\right|\\&\quad\leqslant\|\phi\|_p\left\|k(\mathbf{x}-\cdot\,)-\sum_{\mathbf{j}\leqslant 2\mathbf{n}}k(\mathbf{x}_{\mathbf{j}}-\cdot\,)\,\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x}_{\mathbf{j}})\right\|_{p'},\end{aligned}$$

where $1/p+1/p'=1$, using Hölder's inequality.

Because $1\leqslant p\leqslant 2$, $p'\geqslant 2$, and we can use the Hausdorff–Young Inequality (see [31]) to show that

$$\left\|k(\mathbf{x}-\cdot\,)-\sum_{\mathbf{j}\leqslant 2\mathbf{n}}k(\mathbf{x}_{\mathbf{j}}-\cdot\,)\,\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x}_{\mathbf{j}})\right\|_{p'}\leqslant\left(\sum_{\mathbf{z}\in\mathbb{Z}^d}|b_{\mathbf{z}}|^p\right)^{1/p},$$

where

$$\begin{aligned}b_{\mathbf{z}}&=(2\pi)^{-d}\int_{\mathbb{T}^d}\Big(k(\mathbf{x}-\mathbf{y})-\sum_{\mathbf{j}\leqslant 2\mathbf{n}}k(\mathbf{x}_{\mathbf{j}}-\mathbf{y})\,\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x}_{\mathbf{j}})\Big)\,e^{i\mathbf{z}\mathbf{y}}\,d\mathbf{y}\\&=a_{\mathbf{z}}\left(e^{i\mathbf{z}\mathbf{x}}-\sum_{\mathbf{j}\leqslant 2\mathbf{n}}e^{i\mathbf{z}\mathbf{x}_{\mathbf{j}}}\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x}_{\mathbf{j}})\right),\qquad \mathbf{z}\in\mathbb{Z}^d,\end{aligned}$$

using the fact that the integral above is a sum of convolutions with exponential functions. Thus, using Lemma 20, we see that

$$\begin{aligned}
&\|k(\mathbf{x}-\cdot\,)-\sum_{\mathbf{j}\leqslant 2\mathbf{n}} k(\mathbf{x_j}-\cdot\,)\,\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x_j})\|_{p'}\\
&\quad\leqslant\left(\sum_{\mathbf{z}\in\mathbb{Z}^d} a_{\mathbf{j}}^p\left(\sum_{m=1}^{d}|\theta^m_{n_m,z_m}(x_m)|\right)^p\right)^{1/p}\\
&\quad\leqslant C\sup_{1\leqslant m\leqslant d}\left(\sum_{\ell\geqslant n_m} a^p_{\ell,m}\right)^{1/p}.
\end{aligned}$$

So, if we define the operator $T\colon L_p\to L_\infty$ by

$$T\phi(x)=\int_{\mathbb{T}^d} k(\mathbf{x}-\mathbf{y})\,\phi(\mathbf{y})\,d\mathbf{y}-\int_{\mathbb{T}^d}\left\{\sum_{\mathbf{j}\leqslant 2\mathbf{n}} k(\mathbf{x_j}-\mathbf{y})\,\widetilde{\mathrm{sk}}(\mathbf{x}-\mathbf{x_j})\right\}\phi(\mathbf{y})\,d\mathbf{y},$$

then T is bounded and

$$\|T\|_{p,\infty}\leqslant C\sup_{1\leqslant m\leqslant d}\left(\sum_{\ell\geqslant n_m} a^p_{\ell,m}\right)^{1/p}.$$

Using duality arguments (see for example Tikhomirov [28]) we also have $T\colon L_1\to L_{p'}$ bounded with

$$\|T\|_{1,p'}\leqslant C\sup_{1\leqslant m\leqslant d}\left(\sum_{\ell\geqslant n_m} a^p_{\ell,m}\right)^{1/p}.$$

Applying the Riesz–Thorin Interpolation Theorem [31], for $0<t<1$, if

$$\frac{1}{p_t}=1-t+\frac{1}{p}\qquad\text{and}\qquad\frac{1}{q^t}=\frac{1-t}{p'},$$

then $T\colon L_{p_t}\to L_{q_t}$ is bounded and

$$\|T\|_{p_t,q_t}\leqslant C\sup_{1\leqslant m\leqslant d}\left(\sum_{\ell\geqslant n_m} a^p_{\ell,m}\right)^{1/p}.$$

It is easy to show that $p_t^{-1}-q_t^{-1}=p^{-1}\geqslant 2^{-1}$. Hence, setting $r=p_t$ and $s=q_t$, if $1\leqslant r\leqslant 2\leqslant s\leqslant\infty$ and $r^{-1}-s^{-1}\geqslant 2$ then $T\colon L_r\to L_s$ is bounded and

$$\|T\|_{r,s}\leqslant C\sup_{1\leqslant m\leqslant d}\left(\sum_{\ell\geqslant n_m} a^{sr(s-r)^{-1}}_{\ell,m}\right)^{r^{-1}-s^{-1}}. \quad\blacksquare$$

4. FINAL REMARKS

We conclude by noting that Theorem 21 includes approximation by periodic splines as special cases of the general result. For some kernels sk-spline approximation is the best possible approximation in the sense of n-width. For example, if $d=1$ and $a_\ell = e^{-\alpha \ell^r}$, $\alpha > 0$ and $r \geqslant 1$, then from Theorem 21 it follows that

$$\begin{aligned} \varepsilon_n(k * U_1, L_\infty) &:= \sup \{ \|f - \mathrm{sk}(f, \Delta_n)\|_\infty : f \in k * U_1 \} \\ &\leqslant C e^{-\alpha n^r}, \end{aligned}$$

as $n \to \infty$. Hence, because L_p is continuously imbedded in L_q for $1 \leqslant q \leqslant p \leqslant \infty$, we have $\varepsilon_n(k * U_p, L_q) \leqslant C e^{-\alpha n^r}$, for any $1 \leqslant p, q \leqslant \infty$. It is clear that in this case the dimension of the sk-spline subspace is $2n$. Let d_n and b_n be the n-widths in the sense of Kolmogorov and Bernstein respectively, and λ_n and π_n be, respectively, the linear and projective n-widths (for the definitions of these see e.g. [24, 28]). Since the operator sk: $f \to \mathrm{sk}(f, \Delta_n)$, is a linear projector (due to the uniqueness of the solution of the interpolation problem) we have $b_{2n} \leqslant d_{2n} \leqslant \lambda_{2n} \leqslant \pi_{2n} \leqslant C \varepsilon_{2n}$, where C_3 does not depend on n. Using the results of [11, 12], we have $b_{2n}(k * U_p, L_q) \geqslant C e^{-\alpha n^r}$, as $n \to \infty$, for all $1 \leqslant p, q \leqslant \infty$. So we see that sk-splines provide a new example of extremal subspaces for $k * U_p$ in L_q, for all $1 \leqslant p, q \leqslant \infty$.

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REFERENCES

1. J. H. Ahlberg, E. N. Nielson, and J. L. Walsh, "The Theory of Spline Functions and Their Applications," Academic Press, New York, 1967.
2. P. M. Anselone and P.-J. Laurent, A general method for constructing interpolating or smoothing spline functions, *Numer. Math.* **12** (1968), 66–82.
3. M. Atteia, Généralisation de la définition et des propriétés des "spline functions," *Comp. Rend.* **260** (1965), 3550–3553.
4. N. Dyn, Perfect splines of minimum norm for monotone norms and norms induced by inner products, with applications to tensor product approximation and n-widths of integral operators, *J. Approx. Theory* **38** (1983), 105–138.
5. N. Dyn, F. J. Narcowich, and J. Ward, Variational principles and Sobolev-type estimates for generalised interpolation on Riemannian manifolds, preprint, 1996.

6. V. I. Gorbacuk and M. L. Gorbacuk, Trigonometric series and generalised periodic functions, *Dokl. Akad. Nauk SSSR* **257** (1981), 799–803; *Soviet Math. Dokl.* **23** (1981), 342–346 [English transl.].
7. T. Gutzmer, Interpolation by positive definite functions on compact groups with applications to $SO(3)$, *Results Math.* **29** (1996), 69–77.
8. J. C. Holladay, A smoothest curve approximation, *Math. Comp.* **11** (1957), 233–243.
9. A. K. Kushpel, sk-splines and sharp estimates for the widths of function classes in the space $C_{2\pi}$, *Inst. Mat. Akad. Nauk. Ukrain. SSR* Preprint 85.51, Kiev, 1985. [in Russian]
10. A. K. Kushpel, Sharp estimates of the widths of convolution classes, *Math. USSR Izv.* **33** (1989), 631–649.
11. A. K. Kushpel, n-widths of sets of analytic functions, *Ukrain. Mat. Zh.* **41** (1989), 576–579.
12. A. K. Kushpel, Approximation of smooth functions and n-widths, *Inst. Mat. Acad. Nauk. Ukrain. SSR* Preprint, 89.77 Kiev, 1989. [in Russian]
13. W. A. Light and H. S. J. Wayne, Error estimates for approximation by radial functions, *in* "Approximation Theory, Wavelets and Applications" (S. P. Singh, Ed.), Kluwer, Dordrecht, 1995.
14. L. H. Loomis, "An Introduction to Abstract Harmonic Analysis," Van Nostrand, Princeton, NJ, 1953.
15. W. R. Madych and S. A. Nelson, Multivariate interpolation and conditonally positive definite functions, *Approx. Theory Appl.* **4** (1988), 77–89.
16. W. R. Madych and S. A. Nelson, Multivariate interpolation and conditonally positive definite functions, II, *Math. Comp.* **54** (1990), 211–230.
17. C. A. Micchelli, Interpolation of scattered data: Distance matrices and conditionally positive definite functions, *Constr. Approx.* **2** (1986), 11–22.
18. C. A. Micchelli, Cardinal $\mathscr{L}$-splines, *in* "Studies in Spines and Approximation Theory," Academic Press, San Diego, 1976.
19. C. A. Micchelli and A. Pinkus, On n-widths and optimal recovery in M^r, *in* "Approximation Theory, II" (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), Academic Press, New York, 1976.
20. C. A. Micchelli and A. Pinkus, Total positivity and the exact n-widths of certain sets in L^1, *Pacific J. Math.* **74** (1977), 499–515.
21. C. A. Micchelli and A. Pinkus, On n-widths in L^∞, *Trans. Amer. Math. Soc.* **234** (1977), 139–174.
22. C. A. Micchelli and A. Pinkus, Some problems in the approximation of functions of two variables and n-widths of integral operators, *J. Approx. Theory* **24** (1978), 51–77.
23. F. J. Narcowich, Generalised Hermite interpolation and positive definite kernels on a Riemannian manifold, *J. Math. Anal. Appl.* **190** (1995), 165–193.
24. A. Pinkus, "n-Widths in Approximation Theory," Springer-Verlag, Berlin/New York, 1985.
25. W. Rudin, "Fourier Analysis on Groups," Interscience, New York, 1962.
26. X. Sun, The fundamentality of translates of a continuous function on spheres, *Numer. Algorithms* **8** (1994), 131–134.
27. V. M. Tikhomirov, n-Widths of sets in functional spaces and approximation theory, *Uspekhi Mat. Nauk.* **15** (1960), 81–120. [in Russian]
28. V. M. Tikhomirov, Some questions in approximation theory, *Izdat Moskov. Gos. Univ.*, Moscow, 1976. [in Russian]
29. Z. Wu and R. Schaback, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Numer. Anal.* **13** (1993), 13–27.
30. Y. Xu and E. W. Cheney, Strictly positive definite functions on spheres, *Proc. Amer. Math. Soc.* **4** (1992), 977–981.
31. A. Zygmund, "Trignometric Series, II," Cambridge Univ. Press, Cambridge, UK, 1959.